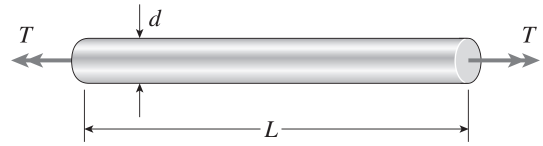
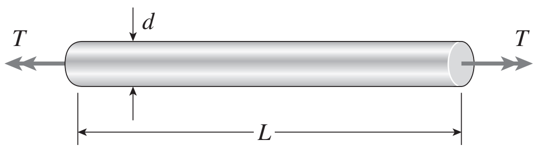
Strain Energy in Torsion

Problem 3.9-1 A solid circular bar of steel ($G = 11.4 \times 10^6$ psi) with length $L = 30$ in. and diameter $d = 1.75$ in. is subjected to pure torsion by torques T acting at the ends (see figure).

(a) Calculate the amount of strain energy U stored in the bar when the maximum shear stress is 4500 psi.
(b) From the strain energy, calculate the angle of twist ϕ (in degrees).



Solution 3.9-1 **Steel bar**



$G = 11.4 \times 10^6$ psi

$L = 30$ in.

$d = 1.75$ in.

$\tau_{max} = 4500$ psi

$$\tau_{max} = \frac{16\,T}{\pi d^3} \qquad T = \frac{\pi d^3 \tau_{max}}{16} \qquad \text{(Eq. 1)}$$

$$I_P = \frac{\pi d^4}{32}$$

(a) Strain energy

$$U = \frac{T^2 L}{2 G I_P} = \left(\frac{\pi d^3 \tau_{max}}{16}\right)^2 \left(\frac{L}{2G}\right)\left(\frac{32}{\pi d^4}\right)$$

$$= \frac{\pi d^2 L \tau_{max}^2}{16G} \qquad \text{(Eq. 2)}$$

Substitute numerical values:

$U = 32.0$ in.-lb $\longleftarrow$

(b) Angle of twist

$$U = \frac{T\phi}{2} \qquad \phi = \frac{2U}{T}$$

Substitute for T and U from Eqs. (1) and (2):

$$\phi = \frac{2L\tau_{max}}{Gd} \qquad \text{(Eq. 3)}$$

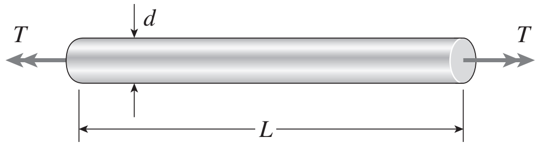
Substitute numerical values:

$\phi = 0.013534$ rad $= 0.775°$ $\longleftarrow$

Problem 3.9-2 A solid circular bar of copper ($G = 45$ GPa) with length $L = 0.75$ m and diameter $d = 40$ mm is subjected to pure torsion by torques T acting at the ends (see figure).

(a) Calculate the amount of strain energy U stored in the bar when the maximum shear stress is 32 MPa.
(b) From the strain energy, calculate the angle of twist ϕ (in degrees)

Solution 3.9-2 Copper bar



$G = 45$ GPa

$L = 0.75$ m

$d = 40$ mm

$\tau_{max} = 32$ MPa

$$\tau_{max} = \frac{16T}{\pi d^3} \quad T = \frac{\pi d^3 \tau_{max}}{16} \qquad \text{(Eq. 1)}$$

$$I_P = \frac{\pi d^4}{32}$$

(A) Strain energy

$$U = \frac{T^2 L}{2GI_P} = \left(\frac{\pi d^3 \tau_{max}}{16}\right)^2 \left(\frac{L}{2G}\right)\left(\frac{32}{\pi d^4}\right)$$

$$= \frac{\pi d^2 L \tau_{max}^2}{16G} \qquad \text{(Eq. 2)}$$

Substitute numerical values:

$U = 5.36$ J ⟵

(b) Angle of twist

$$U = \frac{T\phi}{2} \quad \phi = \frac{2U}{T}$$

Substitute for T and U from Eqs. (1) and (2):

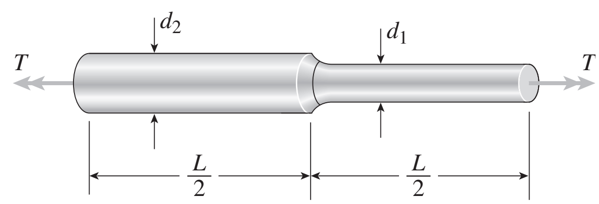
$$\phi = \frac{2L\tau_{max}}{Gd} \qquad \text{(Eq. 3)}$$

Substitute numerical values:

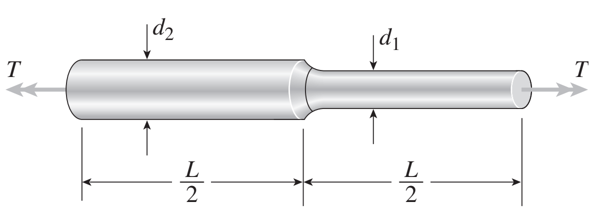
$\phi = 0.026667$ rad $= 1.53°$ ⟵

Problem 3.9-3 A stepped shaft of solid circular cross sections (see figure) has length $L = 45$ in., diameter $d_2 = 1.2$ in., and diameter $d_1 = 1.0$ in. The material is brass with $G = 5.6 \times 10^6$ psi.

Determine the strain energy U of the shaft if the angle of twist is 3.0°.



Solution 3.9-3 Stepped shaft



$d_1 = 1.0$ in.

$d_2 = 1.2$ in.

$L = 45$ in.

$G = 5.6 \times 10^6$ psi (brass)

$\phi = 3.0° = 0.0523599$ rad

Strain energy

$$U = \sum \frac{T^2 L}{2GI_P} = \frac{16\,T^2(L/2)}{\pi G d_2^4} + \frac{16\,T^2(L/2)}{\pi G d_1^4}$$

$$= \frac{8T^2 L}{\pi G}\left(\frac{1}{d_2^4} + \frac{1}{d_1^4}\right) \qquad \text{(Eq. 1)}$$

$$\text{Also, } U = \frac{T\phi}{2} \qquad \text{(Eq. 2)}$$

Equate U from Eqs. (1) and (2) and solve for T:

$$T = \frac{\pi G d_1^4 d_2^4 \phi}{16L(d_1^4 + d_2^4)}$$

$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L}\left(\frac{d_1^4 d_2^4}{d_1^4 + d_2^4}\right) \quad \phi = \text{radians}$$

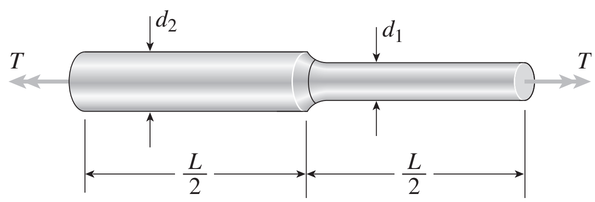
Substitute numerical values:

$U = 22.6$ in.-lb ⟵

Problem 3.9-4 A stepped shaft of solid circular cross sections (see figure) has length $L = 0.80$ m, diameter $d_2 = 40$ mm, and diameter $d_1 = 30$ mm. The material is steel with $G = 80$ GPa.

Determine the strain energy U of the shaft if the angle of twist is 1.0°.

Soluton 3.9-4 Stepped shaft



$d_1 = 30$ mm $\quad d_2 = 40$ mm

$L = 0.80$ m $\quad G = 80$ GPa (steel)

$\phi = 1.0° = 0.0174533$ rad

STRAIN ENERGY

$$U = \sum \frac{T^2 L}{2GI_P} = \frac{16T^2(L/2)}{\pi G d_2^4} + \frac{16T^2(L/2)}{\pi G d_1^4}$$

$$= \frac{8T^2 L}{\pi G}\left(\frac{1}{d_2^4} + \frac{1}{d_1^4}\right) \qquad \text{(Eq. 1)}$$

Also, $U = \dfrac{T\phi}{2}$ (Eq. 2)

Equate U from Eqs. (1) and (2) and solve for T:

$$T = \frac{\pi G\, d_1^4\, d_2^4\, \phi}{16L(d_1^4 + d_2^4)}$$

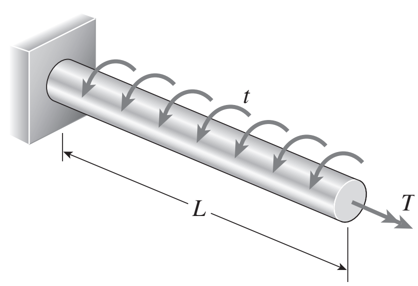
$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L}\left(\frac{d_1^4\, d_2^4}{d_1^4 + d_2^4}\right) \qquad \phi = \text{radians}$$

SUBSTITUTE NUMERICAL VALUES:

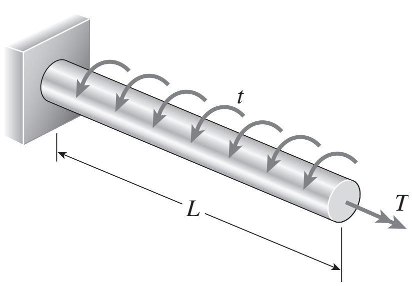
$U = 1.84$ J ⟵

Problem 3.9-5 A cantilever bar of circular cross section and length L is fixed at one end and free at the other (see figure). The bar is loaded by a torque T at the free end and by a distributed torque of constant intensity t per unit distance along the length of the bar.

(a) What is the strain energy U_1 of the bar when the load T acts alone?
(b) What is the strain energy U_2 when the load t acts alone?
(c) What is the strain energy U_3 when both loads act simultaneously?



Solution 3.9-5 Cantilever bar with distributed torque



G = shear modulus

I_P = polar moment of inertia

T = torque acting at free end

t = torque per unit distance

(a) LOAD T ACTS ALONE (EQ. 3-51a)

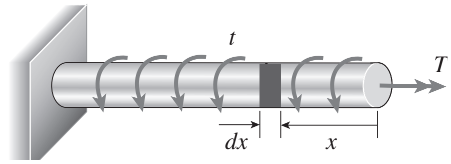
$$U_1 = \frac{T^2L}{2GI_P} \longleftarrow$$

(b) LOAD t ACTS ALONE

From Eq. (3-56) of Example 3-11:

$$U_2 = \frac{t^2L^3}{6GI_P} \longleftarrow$$

(c) BOTH LOADS ACT SIMULTANEOUSLY



At distance x from the free end:

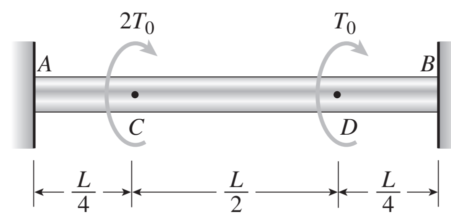
$$T(x) = T + tx$$

$$U_3 = \int_0^L \frac{[T(x)]^2}{2GI_P}\,dx = \frac{1}{2GI_P}\int_0^L (T + tx)^2 dx$$

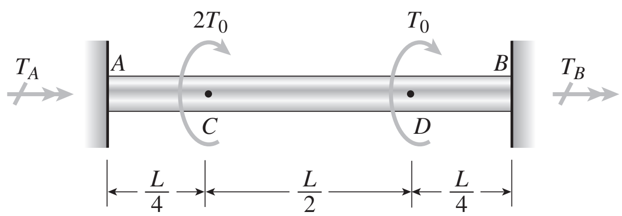
$$= \frac{T^2L}{2GI_P} + \frac{TtL^2}{2GI_P} + \frac{t^2L^3}{6GI_P} \longleftarrow$$

NOTE: U_3 is *not* the sum of U_1 and U_2.

Problem 3.9-6 Obtain a formula for the strain energy U of the statically indeterminate circular bar shown in the figure. The bar has fixed supports at ends A and B and is loaded by torques $2T_0$ and T_0 at points C and D, respectively.
Hint: Use Eqs. 3-46a and b of Example 3-9, Section 3.8, to obtain the reactive torques.



Solution 3.9-6 **Statically indeterminate bar**



REACTIVE TORQUES

From Eq. (3-46a):

$$T_A = \frac{(2T_0)\left(\frac{3L}{4}\right)}{L} + \frac{T_0\left(\frac{L}{4}\right)}{L} = \frac{7T_0}{4}$$

$$T_B = 3T_0 - T_A = \frac{5T_0}{4}$$

INTERNAL TORQUES

$$T_{AC} = -\frac{7T_0}{4} \qquad T_{CD} = \frac{T_0}{4} \qquad T_{DB} = \frac{5T_0}{4}$$

STRAIN ENERGY (from EQ. 3-53)

$$U = \sum_{i=1}^{n} \frac{T_i^2 L_i}{2G_i I_{Pi}}$$

$$= \frac{1}{2GI_p}\left[T_{AC}^2\left(\frac{L}{4}\right) + T_{CD}^2\left(\frac{L}{2}\right) + T_{DB}^2\left(\frac{L}{4}\right)\right]$$

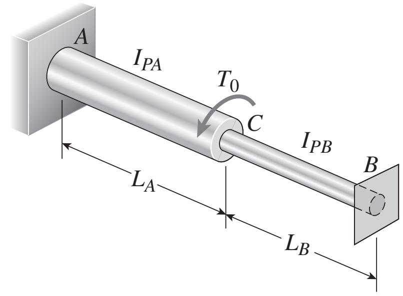
$$= \frac{1}{2GI_P}\left[\left(-\frac{7T_0}{4}\right)^2\left(\frac{L}{4}\right) + \left(\frac{T_0}{4}\right)^2\left(\frac{L}{2}\right) + \left(\frac{5T_0}{4}\right)^2\left(\frac{L}{4}\right)\right]$$

$$U = \frac{19T_0^2L}{32GI_P} \longleftarrow$$

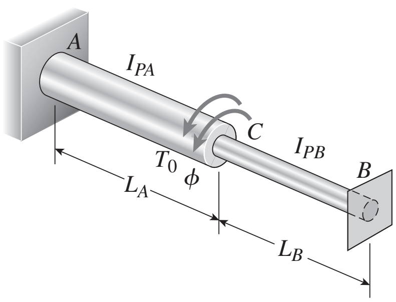
Problem 3.9-7 A statically indeterminate stepped shaft ACB is fixed at ends A and B and loaded by a torque T_0 at point C (see figure). The two segments of the bar are made of the same material, have lengths L_A and L_B, and have polar moments of inertia I_{PA} and I_{PB}.

Determine the angle of rotation ϕ of the cross section at C by using strain energy.

Hint: Use Eq. 3-51b to determine the strain energy U in terms of the angle ϕ. Then equate the strain energy to the work done by the torque T_0. Compare your result with Eq. 3-48 of Example 3-9, Section 3.8.



Solution 3.9-7 **Statically indeterminate bar**



STRAIN ENERGY (FROM EQ. 3-51B)

$$U = \sum_{i=1}^{n} \frac{GI_{Pi}\phi_i^2}{2L_i} = \frac{GI_{PA}\phi^2}{2L_A} + \frac{GI_{PB}\phi^2}{2L_B}$$

$$= \frac{G\phi^2}{2}\left(\frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B}\right)$$

WORK DONE BY THE TORQUE T_0

$$W = \frac{T_0\phi}{2}$$

EQUATE U AND W AND SOLVE FOR ϕ

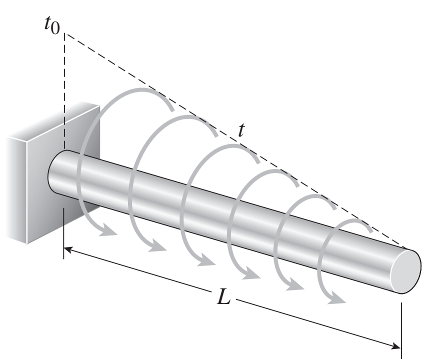
$$\frac{G\phi^2}{2}\left(\frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B}\right) = \frac{T_0\phi}{2}$$

$$\phi = \frac{T_0 L_A L_B}{G(L_B I_{PA} + L_A I_{PB})} \longleftarrow$$

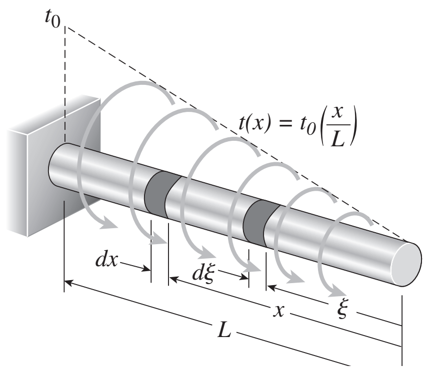
(This result agrees with Eq. (3-48) of Example 3-9, Section 3.8.)

Problem 3.9-8 Derive a formula for the strain energy U of the cantilever bar shown in the figure.

The bar has circular cross sections and length L. It is subjected to a distributed torque of intensity t per unit distance. The intensity varies linearly from $t = 0$ at the free end to a maximum value $t = t_0$ at the support.



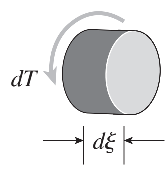
Solution 3.9-8 Cantilever bar with distributed torque



x = distance from right-hand end of the bar

Element $d\xi$

Consider a differential element $d\xi$ at distance ξ from the right-hand end.

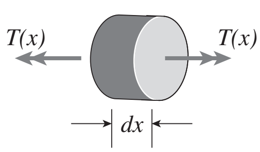


dT = external torque acting on this element

$$dT = t(\xi)d\xi$$

$$= t_0\left(\frac{\xi}{L}\right)d\xi$$

Element dx at distance x



$T(x)$ = internal torque acting on this element

$T(x)$ = total torque from $x = 0$ to $x = x$

$$T(x) = \int_0^x dT = \int_0^x t_0\left(\frac{\xi}{L}\right)d\xi$$

$$= \frac{t_0 x^2}{2L}$$

Strain energy of element dx

$$dU = \frac{[T(x)]^2 dx}{2GI_P} = \frac{1}{2GI_P}\left(\frac{t_0}{2L}\right)^2 x^4\,dx$$

$$= \frac{t_0^2}{8L^2GI_P}x^4\,dx$$

Strain energy of entire bar

$$U = \int_0^L dU = \frac{t_0^2}{8L^2GI_P}\int_0^L x^4\,dx$$

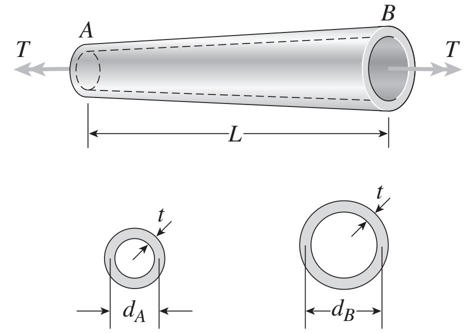
$$= \frac{t_0^2}{8L^2GI_P}\left(\frac{L^5}{5}\right)$$

$$U = \frac{t_0^2L^3}{40GI_P} \longleftarrow$$

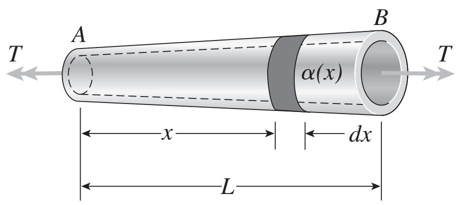
Problem 3.9-9 A thin-walled hollow tube *AB* of conical shape has constant thickness *t* and average diameters d_A and d_B at the ends (see figure).

(a) Determine the strain energy *U* of the tube when it is subjected to pure torsion by torques *T*.
(b) Determine the angle of twist ϕ of the tube.

Note: Use the approximate formula $I_P \approx \pi d^3 t/4$ for a thin circular ring; see Case 22 of Appendix D.



Solution 3.9-9 Thin-walled, hollow tube



t = thickness

d_A = average diameter at end *A*

d_B = average diameter at end *B*

$d(x)$ = average diameter at distance *x* from end *A*

$$d(x) = d_A + \left(\frac{d_B - d_A}{L}\right)x$$

POLAR MOMENT OF INERTIA

$$I_P = \frac{\pi d^3 t}{4}$$

$$I_P(x) = \frac{\pi[d(x)]^3 t}{4} = \frac{\pi t}{4}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3$$

(a) STRAIN ENERGY (FROM EQ. 3-54)

$$U = \int_0^L \frac{T^2 dx}{2GI_P(x)}$$

$$= \frac{2T^2}{\pi G t}\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3} \qquad \text{(Eq. 1)}$$

From Appendix C:

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

Therefore,

$$\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

$$= -\frac{1}{\frac{2(d_B - d_A)}{L}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^2}\Bigg|_0^L$$

$$= -\frac{L}{2(d_B - d_A)(d_B)^2} + \frac{L}{2(d_B - d_A)(d_A)^2}$$

$$= \frac{L(d_A + d_B)}{2d_A^2\, d_B^2}$$

Substitute this expression for the integral into the equation for *U* (Eq. 1):

$$U = \frac{2T^2}{\pi G t} \cdot \frac{L(d_A + d_B)}{2d_A^2 d_B^2} = \frac{T^2 L}{\pi G t}\left(\frac{d_A + d_B}{d_A^2\, d_B^2}\right) \longleftarrow$$

(b) ANGLE OF TWIST

Work of the torque *T*: $W = \frac{T\phi}{2}$

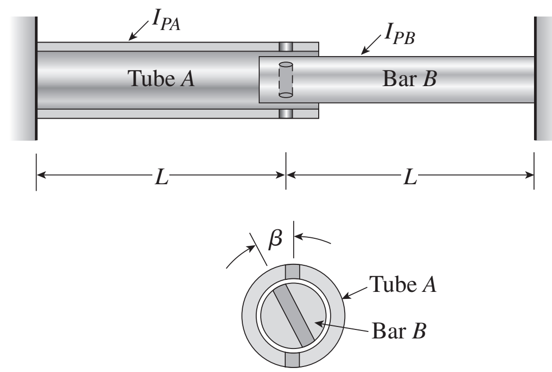
$$W = U \qquad \frac{T\phi}{2} = \frac{T^2 L(d_A + d_B)}{\pi G t\, d_A^2 d_B^2}$$

Solve for ϕ:

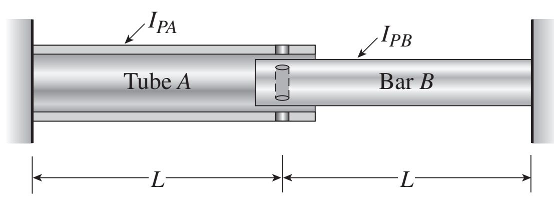
$$\phi = \frac{2TL(d_A + d_B)}{\pi G t\, d_A^2 d_B^2} \longleftarrow$$

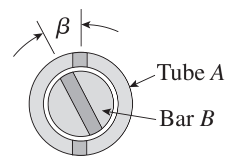
Problem 3.9-10 A hollow circular tube A fits over the end of a solid circular bar B, as shown in the figure. The far ends of both bars are fixed. Initially, a hole through bar B makes an angle β with a line through two holes in tube A. Then bar B is twisted until the holes are aligned, and a pin is placed through the holes.

When bar B is released and the system returns to equilibrium, what is the total strain energy U of the two bars? (Let I_{PA} and I_{PB} represent the polar moments of inertia of bars A and B, respectively. The length L and shear modulus of elasticity G are the same for both bars.)

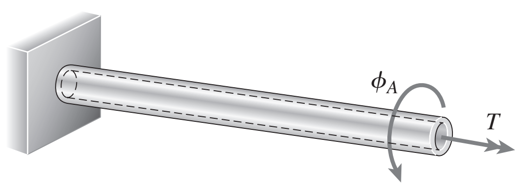


Solution 3.9-10 **Circular tube and bar**





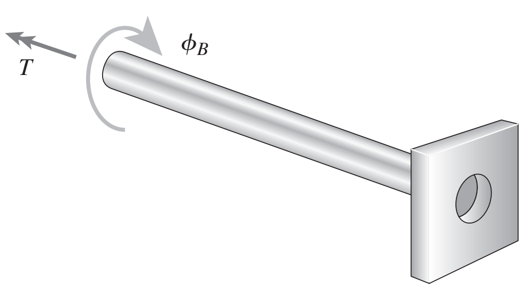
Tube A



T = torque acting on the tube

ϕ_A = angle of twist

Bar B



T = torque acting on the bar

ϕ_B = angle of twist

Compatibility

$$\phi_A + \phi_B = \beta$$

Force-displacement relations

$$\phi_A = \frac{TL}{GI_{PA}} \qquad \phi_B = \frac{TL}{GI_{PB}}$$

Substitute into the equation of compatibility and solve for T:

$$T = \frac{\beta G}{L}\left(\frac{I_{PA}I_{PB}}{I_{PA} + I_{PB}}\right)$$

Strain energy

$$U = \sum \frac{T^2L}{2GI_P} = \frac{T^2L}{2GI_{PA}} + \frac{T^2L}{2GI_{PB}}$$

$$= \frac{T^2L}{2G}\left(\frac{1}{I_{PA}} + \frac{1}{I_{PB}}\right)$$

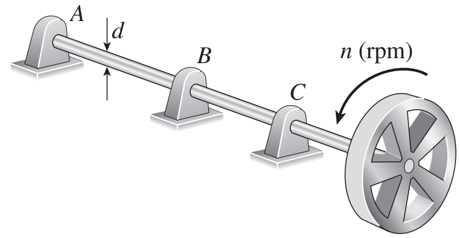
Substitute for T and simplify:

$$U = \frac{\beta^2 G}{2L}\left(\frac{I_{PA}\,I_{PB}}{I_{PA} + I_{PB}}\right) \longleftarrow$$

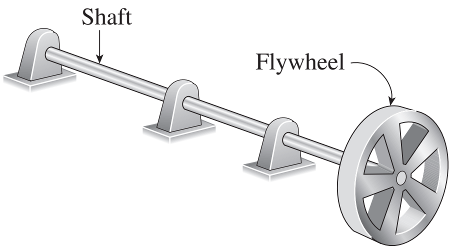
****Problem 3.9-11** A heavy flywheel rotating at n revolutions per minute is rigidly attached to the end of a shaft of diameter d (see figure). If the bearing at A suddenly freezes, what will be the maximum angle of twist ϕ of the shaft? What is the corresponding maximum shear stress in the shaft?

(Let L = length of the shaft, G = shear modulus of elasticity, and I_m = mass moment of inertia of the flywheel about the axis of the shaft. Also, disregard friction in the bearings at B and C and disregard the mass of the shaft.)

Hint: Equate the kinetic energy of the rotating flywheel to the strain energy of the shaft.



Solution 3.9-11 Rotating flywheel



d = diameter

n = rpm

KINETIC ENERGY OF FLYWHEEL

$$\text{K.E.} = \frac{1}{2} I_m \omega^2$$

$$\omega = \frac{2\pi n}{60}$$

n = rpm

$$\text{K.E.} = \frac{1}{2} I_m \left(\frac{2\pi n}{60}\right)^2 = \frac{\pi^2 n^2 I_m}{1800}$$

UNITS:

I_m = (force)(length)(second)^2

ω = radians per second

K.E. = (length)(force)

STRAIN ENERGY OF SHAFT (FROM EQ. 3-51b)

$$U = \frac{G I_P \phi^2}{2L}$$

$$I_P = \frac{\pi}{32} d^4$$

d = diameter of shaft

$$U = \frac{\pi G d^4 \phi^2}{64L}$$

UNITS:

G = (force)/(length)^2

I_P = (length)^4

ϕ = radians

L = length

U = (length)(force)

EQUATE KINETIC ENERGY AND STRAIN ENERGY

$$\text{K.E.} = U \quad \frac{\pi^2 n^2 I_m}{1800} = \frac{\pi G d^4 \phi^2}{64L}$$

Solve for ϕ:

$$\phi = \frac{2n}{15d^2}\sqrt{\frac{2\pi I_m L}{G}} \longleftarrow$$

MAXIMUM SHEAR STRESS

$$\tau = \frac{T(d/2)}{I_P} \qquad \phi = \frac{TL}{GI_P}$$

Eliminate T:

$$\tau = \frac{Gd\phi}{2L}$$

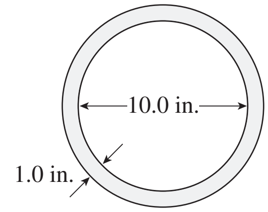
$$\tau_{\max} = \frac{Gd}{2L} \cdot \frac{2n}{15d^2}\sqrt{\frac{2\pi I_m L}{G}}$$

$$\tau_{\max} = \frac{n}{15d}\sqrt{\frac{2\pi G I_m}{L}} \longleftarrow$$

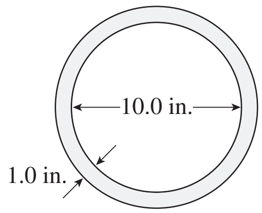
Thin-Walled Tubes

Problem 3.10-1 A hollow circular tube having an inside diameter of 10.0 in. and a wall thickness of 1.0 in. (see figure) is subjected to a torque $T = 1200$ k-in.

Determine the maximum shear stress in the tube using (a) the approximate theory of thin-walled tubes, and (b) the exact torsion theory. Does the approximate theory give conservative or nonconservative results?



Solution 3.10-1 **Hollow circular tube**



$T = 1200$ k-in.

$t = 1.0$ in.

r = radius to median line

$r = 5.5$ in.

d_2 = outside diameter = 12.0 in.

d_1 = inside diameter = 10.0 in.

APPROXIMATE THEORY (EQ. 3-63)

$$\tau_1 = \frac{T}{2\pi r^2 t} = \frac{1200 \text{ k-in.}}{2\pi(5.5 \text{ in.})^2(1.0 \text{ in.})} = 6314 \text{ psi}$$

$$\tau_{approx} = 6310 \text{ psi} \longleftarrow$$

EXACT THEORY (EQ. 3-11)

$$\tau_2 = \frac{T(d_2/2)}{I_P} = \frac{Td_2}{2\left(\frac{\pi}{32}\right) d_2^4 - d_1^4}$$

$$= \frac{16(1200 \text{ k-in.})(12.0 \text{ in.})}{\pi[(12.0 \text{ in.})^4 - (10.0 \text{ in.})^4]}$$

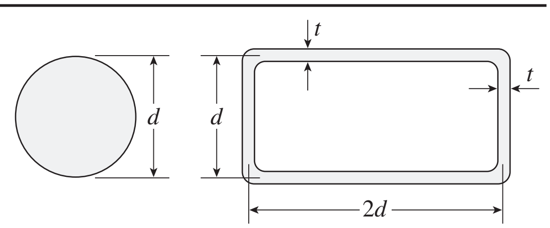
$$= 6831 \text{ psi}$$

$$\tau_{exact} = 6830 \text{ psi} \longleftarrow$$

Because the approximate theory gives stresses that are too low, it is nonconservative. Therefore, the approximate theory should only be used for very thin tubes.

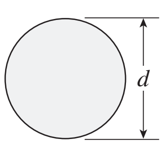
Problem 3.10-2 A solid circular bar having diameter d is to be replaced by a rectangular tube having cross-sectional dimensions $d \times 2d$ to the median line of the cross section (see figure).

Determine the required thickness t_{min} of the tube so that the maximum shear stress in the tube will not exceed the maximum shear stress in the solid bar.



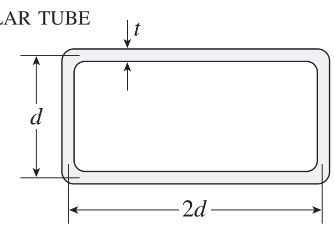
Solution 3.10-2 **Bar and tube**

SOLID BAR



$$\tau_{max} = \frac{16T}{\pi d^3} \quad \text{(Eq. 3-12)}$$

RECTANGULAR TUBE



$$A_m = (d)(2d) = 2d^2 \quad \text{(Eq. 3-64)}$$

$$\tau_{max} = \frac{T}{2tA_m} = \frac{T}{4td^2} \quad \text{(Eq. 3-61)}$$

EQUATE THE MAXIMUM SHEAR STRESSES AND SOLVE FOR t

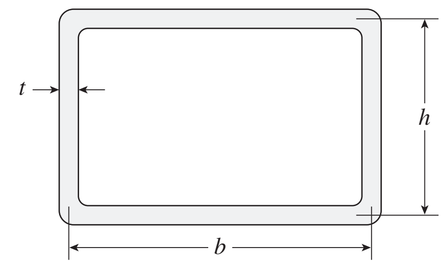
$$\frac{16T}{\pi d^3} = \frac{T}{4td^2}$$

$$t_{min} = \frac{\pi d}{64} \longleftarrow$$

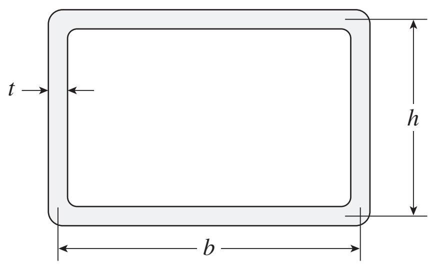
If $t > t_{min}$, the shear stress in the tube is less than the shear stress in the bar.

Problem 3.10-3 A thin-walled aluminum tube of rectangular cross section (see figure) has a centerline dimensions $b = 6.0$ in. and $h = 4.0$ in. The wall thickness t is constant and equal to 0.25 in.

(a) Determine the shear stress in the tube due to a torque $T = 15$ k-in.
(b) Determine the angle of twist (in degrees) if the length L of the tube is 50 in. and the shear modulus G is 4.0×10^6 psi.



Solution 3.10-3 **Thin-walled tube**



$b = 6.0$ in.

$h = 4.0$ in.

$t = 0.25$ in.

$T = 15$ k-in.

$L = 50$ in.

$G = 4.0 \times 10^6$ psi

Eq. (3-64): $A_m = bh = 24.0 \text{ in.}^2$

Eq. (3-71) with $t_1 = t_2 = t$: $J = \dfrac{2b^2h^2t}{b + h}$

$J = 28.8 \text{ in.}^4$

(a) Shear stress (Eq. 3-61)

$$\tau = \frac{T}{2tA_m} = 1250 \text{ psi} \longleftarrow$$

(b) Angle of twist (Eq. 3-72)

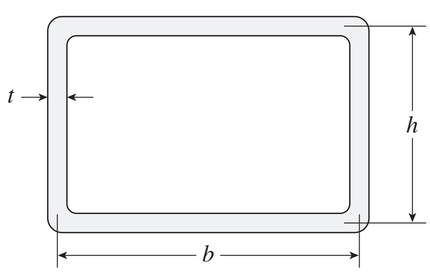
$$\phi = \frac{TL}{GJ} = 0.0065104 \text{ rad}$$

$$= 0.373°$$

Problem 3.10-4 A thin-walled steel tube of rectangular cross section (see figure) has centerline dimensions $b = 150$ mm and $h = 100$ mm. The wall thickness t is constant and equal to 6.0 mm.

(a) Determine the shear stress in the tube due to a torque $T = 1650 \text{ N} \cdot \text{m}$.
(b) Determine the angle of twist (in degrees) if the length L of the tube is 1.2 m and the shear modulus G is 75 GPa.

Solution 3.10-4 **Thin-walled tube**



$b = 150$ mm

$h = 100$ mm

$t = 6.0$ mm

$T = 1650 \text{ N} \cdot \text{m}$

$L = 1.2$ m

$G = 75$ GPa

Eq. (3-64): $A_m = bh = 0.015 \text{ m}^2$

Eq. (3-71) with $t_1 = t_2 = t$: $J = \dfrac{2b^2h^2t}{b + h}$

$J = 10.8 \times 10^{-6} \text{ m}^4$

(a) Shear stress (Eq. 3-61)

$$\tau = \frac{T}{2tA_m} = 9.17 \text{ MPa} \longleftarrow$$

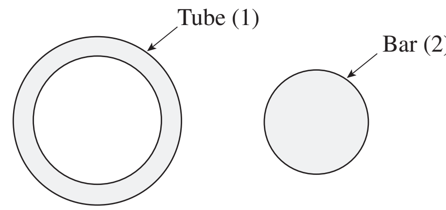
(b) Angle of twist (Eq. 3-72)

$$\phi = \frac{TL}{GJ} = 0.002444 \text{ rad}$$

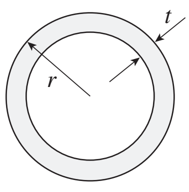
$$= 0.140° \longleftarrow$$

Problem 3.10-5 A thin-walled circular tube and a solid circular bar of the same material (see figure) are subjected to torsion. The tube and bar have the same cross-sectional area and the same length.

What is the ratio of the strain energy U_1 in the tube to the strain energy U_2 in the solid bar if the maximum shear stresses are the same in both cases? (For the tube, use the approximate theory for thin-walled bars.)



Solution 3.10-5 **Thin-walled tube (1)**



$$A_m = \pi r^2 \quad J = 2\pi r^3 t \quad A = 2\pi r t$$

$$\tau_{max} = \frac{T}{2tA_m} = \frac{T}{2\pi r^2 t}$$

$$T = 2\pi r^2 t\tau_{max}$$

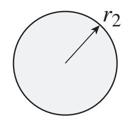
$$U_1 = \frac{T^2 L}{2GJ} = \frac{(2\pi r^2 t\tau_{max})^2 L}{2G(2\pi r^3 t)}$$

$$= \frac{\pi r t \tau_{max}^2 L}{G}$$

But $rt = \dfrac{A}{2\pi}$

$$\therefore\ U_1 = \frac{A\tau_{max}^2 L}{2G}$$

SOLID BAR (2)



$$A = \pi r_2^2 \quad I_P = \frac{\pi}{2} r_2^4$$

$$\tau_{max} = \frac{Tr_2}{I_P} = \frac{2T}{\pi r_2^3} \qquad T = \frac{\pi r_2^3 \tau_{max}}{2}$$

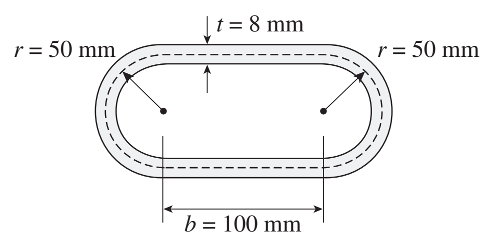
$$U_2 = \frac{T^2 L}{2GI_P} = \frac{(\pi r_2^3 \tau_{max})^2 L}{8G\left(\frac{\pi}{2} r_2^4\right)} = \frac{\pi r_2^2 \tau_{max}^2 L}{4G}$$

But $\pi r_2^2 = A$ $\quad \therefore\ U_2 = \dfrac{A\tau_{max}^2 L}{4G}$

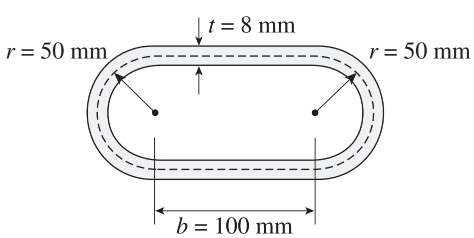
RATIO

$$\frac{U_1}{U_2} = 2 \longleftarrow$$

Problem 3.10-6 Calculate the shear stress τ and the angle of twist ϕ (in degrees) for a steel tube (G = 76 GPa) having the cross section shown in the figure. The tube has length L = 1.5 m and is subjected to a torque T = 10 kN · m.



Solution 3.10-6 **Steel tube**



G = 76 GPa

L = 1.5 m

T = 10 kN · m

$$A_m = \pi r^2 + 2br$$

$$= \pi(50\text{ mm})^2 + 2(100\text{ mm})(50\text{ mm})$$

$$= 17{,}850\text{ mm}^2$$

$$L_m = 2b + 2\pi r$$

$$= 2(100\text{ mm}) + 2\pi(50\text{ mm})$$

$$= 514.2\text{ mm}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(8\text{ mm})(17{,}850\text{ mm}^2)^2}{514.2\text{ mm}}$$

$$= 19.83 \times 10^6\text{ mm}^4$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{10\text{ k}N\cdot\text{m}}{2(8\text{ mm})(17{,}850\text{ mm}^2)}$$

$$= 35.0\text{ MPa} \longleftarrow$$

ANGLE OF TWIST

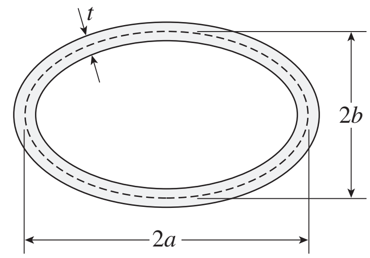
$$\phi = \frac{TL}{GJ} = \frac{(10\text{ k}N\cdot\text{m})(1.5\text{ m})}{(76\text{ GPa})(19.83 \times 10^6\text{ mm}^4)}$$

$$= 0.00995\text{ rad}$$

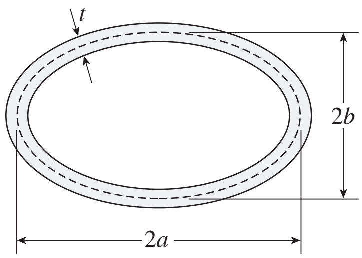
$$= 0.570° \longleftarrow$$

Problem 3.10-7 A thin-walled steel tube having an elliptical cross section with constant thickness t (see figure) is subjected to a torque $T = 18$ k-in.

Determine the shear stress τ and the rate of twist θ (in degrees per inch) if $G = 12 \times 10^6$ psi, $t = 0.2$ in., $a = 3$ in., and $b = 2$ in. (*Note:* See Appendix D, Case 16, for the properties of an ellipse.)



Solution 3.10-7 **Elliptical tube**



$T = 18$ k-in.

$G = 12 \times 10^6$ psi

$t =$ constant

$t = 0.2$ in $a = 3.0$ in. $b = 2.0$ in.

FROM APPENDIX D, CASE 16:

$$A_m = \pi ab = \pi(3.0 \text{ in.})(2.0 \text{ in.}) = 18.850 \text{ in.}^2$$

$$L_m \approx \pi[1.5(a + b) - \sqrt{ab}]$$

$$= \pi[1.5(5.0 \text{ in.}) - \sqrt{6.0 \text{ in.}^2}] = 15.867 \text{ in.}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(0.2 \text{ in.})(18.850 \text{ in.}^2)^2}{15.867 \text{ in.}}$$

$$= 17.92 \text{ in.}^4$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{18 \text{ k} - \text{in.}}{2(0.2 \text{ in.})(18.850 \text{ in.}^2)}$$

$$= 2390 \text{ psi} \longleftarrow$$

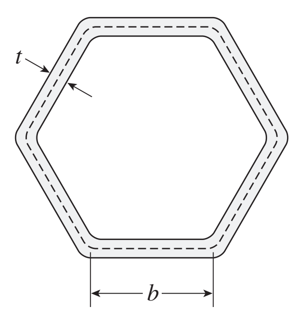
ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

$$\theta = \frac{\phi}{L} = \frac{T}{GJ} = \frac{18 \text{ k-in.}}{(12 \times 10^6 \text{ psi})(17.92 \text{ in.}^4)}$$

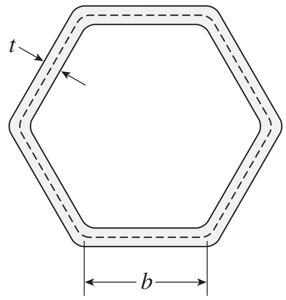
$$\theta = 83.73 \times 10^{-6} \text{ rad/in.} = 0.0048°/\text{in.} \longleftarrow$$

Problem 3.10-8 A torque T is applied to a thin-walled tube having a cross section in the shape of a regular hexagon with constant wall thickness t and side length b (see figure).

Obtain formulas for the shear stress τ and the rate of twist θ.



Solution 3.10-8 **Regular hexagon**



b = Length of side

t = Thickness

$L_m = 6b$

FROM APPENDIX D, CASE 25:

$\beta = 60° \quad n = 6$

$$A_m = \frac{nb^2}{4}\cot\frac{\beta}{2} = \frac{6b^2}{4}\cot 30°$$

$$= \frac{3\sqrt{3}b^2}{2}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{T\sqrt{3}}{9b^2t} \longleftarrow$$

ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

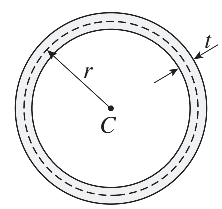
$$J = \frac{4A_m^2}{\int_0^{L_m}\frac{ds}{t}} = \frac{4A_m^2 t}{L_m} = \frac{9b^3t}{2}$$

$$\theta = \frac{T}{GJ} = \frac{2T}{G(9b^3t)} = \frac{2T}{9Gb^3t} \longleftarrow$$

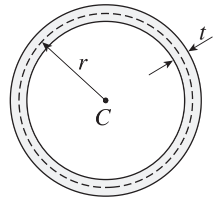
(radians per unit length)

Problem 3.10-9 Compare the angle of twist ϕ_1 for a thin-walled circular tube (see figure) calculated from the approximate theory for thin-walled bars with the angle of twist ϕ_2 calculated from the exact theory of torsion for circular bars.

(a) Express the ratio ϕ_1/ϕ_2 in terms of the nondimensional ratio $\beta = r/t$.
(b) Calculate the ratio of angles of twist for $\beta = 5$, 10, and 20. What conclusion about the accuracy of the approximate theory do you draw from these results?



Solution 3.10-9 **Thin-walled tube**



APPROXIMATE THEORY

$$\phi_1 = \frac{TL}{GJ} \quad J = 2\pi r^3 t \quad \phi_1 = \frac{TL}{2\pi G r^3 t}$$

EXACT THEORY

$$\phi_2 = \frac{TL}{GI_P} \quad \text{From Eq. (3-17): } I_p = \frac{\pi rt}{2}(4r^2 + t^2)$$

$$\phi_2 = \frac{TL}{GI_P} = \frac{2TL}{\pi G r t(4r^2 + t^2)}$$

RATIO

$$\frac{\phi_1}{\phi_2} = \frac{4r^2 + t^2}{4r^2} = 1 + \frac{t^2}{4r^2}$$

$$\text{Let } \beta = \frac{r}{t} \quad \frac{\phi_1}{\phi_2} = 1 + \frac{1}{4\beta^2} \longleftarrow$$

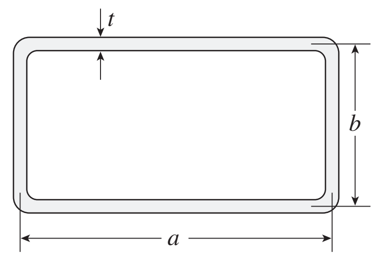
β	ϕ_1/ϕ_2
5	1.0100
10	1.0025
20	1.0006

As the tube becomes thinner and β becomes larger, the ratio ϕ_1/ϕ_2 approaches unity. Thus, the thinner the tube, the more accurate the approximate theory becomes.

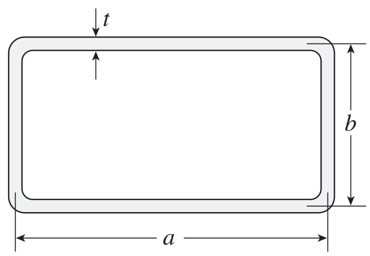
***Problem 3.10-10** A thin-walled rectangular tube has uniform thickness t and dimensions $a \times b$ to the median line of the cross section (see figure).

How does the shear stress in the tube vary with the ratio $\beta = a/b$ if the total length L_m of the median line of the cross section and the torque T remain constant?

From your results, show that the shear stress is smallest when the tube is square ($\beta = 1$).



Solution 3.10-10 **Rectangular tube**



t = thickness (constant)

a, b = Dimensions of the tube

$$\beta = \frac{a}{b}$$

$$L_m = 2(a + b) = \text{constant}$$

$$T = \text{constant}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \qquad A_m = ab = \beta b^2$$

$$L_m = 2b(1 + \beta) = \text{constant}$$

$$b = \frac{L_m}{2(1 + \beta)} \qquad A_m = \beta\left[\frac{L_m}{2(1 + \beta)}\right]^2$$

$$= \frac{\beta L_m^2}{4(1 + \beta)^2}$$

$$\tau = \frac{T}{2tA_m} = \frac{T(4)(1 + \beta)^2}{2t\beta L_m^2} = \frac{2T(1 + \beta)^2}{tL_m^2\beta} \longleftarrow$$

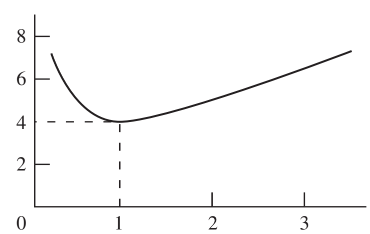
T, t, and L_m are constants.

$$\text{Let } k = \frac{2T}{tL_m^2} = \text{constant} \qquad \tau = k\frac{(1 + \beta)^2}{\beta}$$

$$\left(\frac{\tau}{k}\right)_{\min} = 4$$

$$\tau_{\min} = \frac{8T}{tL_m^2}$$

$$\beta = \frac{a}{b}$$



From the graph, we see that τ is minimum when $\beta = 1$ and the tube is square.

ALTERNATE SOLUTION

$$\tau = \frac{2T}{tL_m^2}\left[\frac{(1 + \beta)^2}{\beta}\right]$$

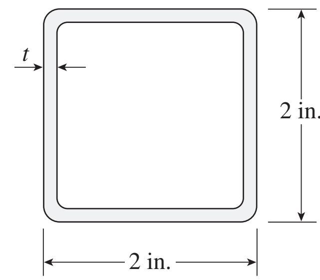
$$\frac{d\tau}{d\beta} = \frac{2T}{tL_m^2}\left[\frac{\beta(2)(1 + \beta) - (1 + \beta)^2(1)}{\beta^2}\right] = 0$$

or $2\beta\,(1+\beta)-(1+\beta)^2 = 0 \quad \therefore \beta = 1$

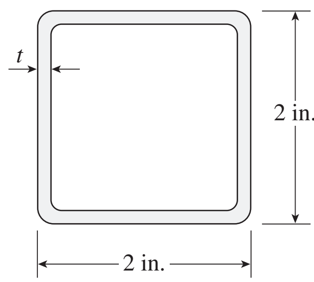
Thus, the tube is square and τ is either a minimum or a maximum. From the graph, we see that τ is a minimum.

***Problem 3.10-11** A tubular aluminum bar ($G = 4 \times 10^6$ psi) of square cross section (see figure) with outer dimensions 2 in. × 2 in. must resist a torque $T = 3000$ lb-in.

Calculate the minimum required wall thickness t_{min} if the allowable shear stress is 4500 psi and the allowable rate of twist is 0.01 rad/ft.



Solution 3.10-11 Square aluminum tube



Outer dimensions:

2.0 in. × 2.0 in.

$G = 4 \times 10^6$ psi

$T = 3000$ lb-in.

$\tau_{allow} = 4500$ psi

$$\theta_{allow} = 0.01 \text{ rad/ft} = \frac{0.01}{12} \text{ rad/in.}$$

Let b = outer dimension

= 2.0 in.

Centerline dimension = $b - t$

$A_m = (b - t)^2 \qquad L_m = 4(b - t)$

$$J = \frac{4tA_m^2}{L_m} = \frac{4t(b-t)^4}{4(b-t)} = t(b-t)^3$$

THICKNESS t BASED UPON SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \qquad tA_m = \frac{T}{2\tau} \qquad t(b-t)^2 = \frac{T}{2\tau}$$

UNITS: t = in. b = in. T = lb-in. τ = psi

$$t(2.0 \text{ in.} - t)^2 = \frac{3000 \text{ lb-in.}}{2(4500 \text{ psi})} = \frac{1}{3} \text{ in.}^3$$

$$3t(2-t)^2 - 1 = 0$$

Solve for t: $t = 0.0915$ in.

THICKNESS t BASED UPON RATE OF TWIST

$$\theta = \frac{T}{GJ} = \frac{T}{Gt(b-t)^3} \qquad t(b-t)^3 = \frac{T}{G\theta}$$

UNITS: t = in. G = psi θ = rad/in.

$$t(2.0 \text{ in.} - t)^3 = \frac{3000 \text{ lb-in}}{(4 \times 10^6 \text{ psi})(0.01/12 \text{ rad/in.})} = \frac{9}{10}$$

$$10t(2-t)^3 - 9 = 0$$

Solve for t:

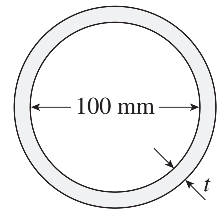
$t = 0.140$ in.

ANGLE OF TWIST GOVERNS

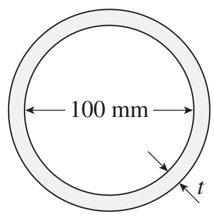
$t_{min} = 0.140$ in. ⟵

***Problem 3.10-12** A thin tubular shaft of circular cross section (see figure) with inside diameter 100 mm is subjected to a torque of 5000 N · m.

If the allowable shear stress is 42 MPa, determine the required wall thickness t by using (a) the approximate theory for a thin-walled tube, and (b) the exact torsion theory for a circular bar.



Solution 3.10-12 Thin tube



$T = 5{,}000\ \text{N}\cdot\text{m}$ $\quad d_1$ = inner diameter = 100 mm

$\tau_{\text{allow}} = 42\ \text{MPa}$

t is in millimeters.

r = Average radius

$= 50\ \text{mm} + \dfrac{t}{2}$

r_1 = Inner radius

= 50 mm

r_2 = Outer radius $\quad A_m = \pi r^2$

= 50 mm + t

(a) APPROXIMATE THEORY

$$\tau = \frac{T}{2tA_m} = \frac{T}{2t(\pi r^2)} = \frac{T}{2\pi r^2 t}$$

$$42\ \text{MPa} = \frac{5{,}000\ \text{N}\cdot\text{m}}{2\pi\left(50 + \frac{t}{2}\right)^2 t}$$

or

$$t\left(50 + \frac{t}{2}\right)^2 = \frac{5{,}000\ \text{N}\cdot\text{m}}{2\pi(42\ \text{MPa})} = \frac{5 \times 10^6}{84\pi}\ \text{mm}^3$$

Solve for t:

$t = 6.66$ mm ⟵

(b) EXACT THEORY

$$\tau = \frac{Tr_2}{I_p} \qquad I_p = \frac{\pi}{2}(r_2^4 - r_1^4)$$

$$= \frac{\pi}{2}[(50 + t)^4 - (50)^4]$$

$$42\ \text{MPa} = \frac{(5{,}000\ \text{N}\cdot\text{m})(50 + t)}{\frac{\pi}{2}[(50 + t)^4 - (50)^4]}$$

$$\frac{(50 + t)^4 - (50)^4}{50 + t} = \frac{(5000\ \text{N}\cdot\text{m})(2)}{(\pi)(42\ \text{MPa})}$$

$$= \frac{5 \times 10^6}{21\pi}\ \text{mm}^3$$

Solve for t:

$t = 7.02$ mm ⟵

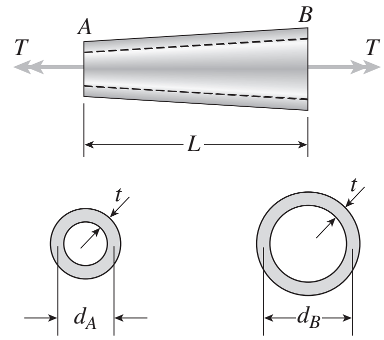
The approximate result is 5% less than the exact result. Thus, the approximate theory is nonconservative and should only be used for thin-walled tubes.

••Problem 3.10-13 A long, thin-walled tapered tube *AB* of circular cross section (see figure) is subjected to a torque *T*. The tube has length *L* and constant wall thickness *t*. The diameter to the median lines of the cross sections at the ends *A* and *B* are d_A and d_B, respectively.

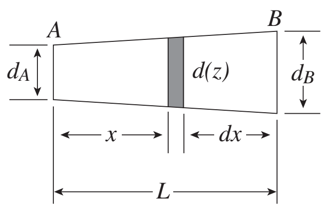
Derive the following formula for the angle of twist of the tube:

$$\phi = \frac{2TL}{\pi G t}\left(\frac{d_A + d_B}{d_A^2 d_B^2}\right)$$

Hint: If the angle of taper is small, we may obtain approximate results by applying the formulas for a thin-walled prismatic tube to a differential element of the tapered tube and then integrating along the axis of the tube.



Solution 3.10-13 **Thin-walled tapered tube**



t = thickness

d_A = average diameter at end *A*

d_B = average diameter at end *B*

T = torque

$d(x)$ = average diameter at distance *x* from end *A*.

$$d(x) = d_A + \left(\frac{d_B - d_A}{L}\right)x$$

$$J = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

$$J(x) = \frac{\pi t}{4}[d(x)]^3 = \frac{\pi t}{4}\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3$$

For element of length *dx*:

$$d\phi = \frac{Tdx}{GJ(x)} = \frac{4Tdx}{G\pi t\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

For entire tube:

$$\phi = \frac{4T}{\pi GT}\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L}\right)x\right]^3}$$

From table of integrals (see Appendix C):

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

$$\phi = \frac{4T}{\pi Gt}\left[-\frac{1}{2\left(\frac{d_B - d_A}{L}\right)\left(d_A + \frac{d_B - d_A}{L}\cdot x\right)^2}\right]_0^L$$

$$= \frac{4T}{\pi Gt}\left[-\frac{L}{2(d_B - d_A)d_B^2} + \frac{L}{2(d_B - d_A)d_A^2}\right]$$

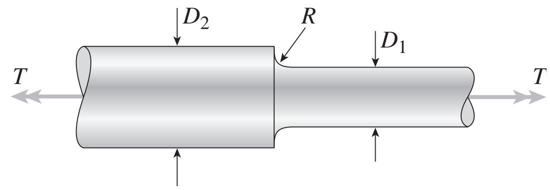
$$\phi = \frac{2TL}{\pi Gt}\left(\frac{d_A + d_B}{d_A^2 d_B^2}\right) \longleftarrow$$

Stress Concentrations in Torsion

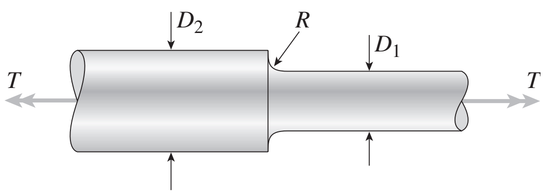
The problems for Section 3.11 are to be solved by considering the stress-concentration factors.

Problem 3.11-1 A stepped shaft consisting of solid circular segments having diameters $D_1 = 2.0$ in. and $D_2 = 2.4$ in. (see figure) is subjected to torques T. The radius of the fillet is $R = 0.1$ in.

If the allowable shear stress at the stress concentration is 6000 psi, what is the maximum permissible torque T_{max}?



Solution 3.11-1 **Stepped shaft in torsion**



$D_1 = 2.0$ in.

$D_2 = 2.4$ in.

$R = 0.1$ in.

$\tau_{allow} = 6000$ psi

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\frac{R}{D_1} = \frac{0.1 \text{ in.}}{2.0 \text{ in.}} = 0.05 \qquad \frac{D_2}{D_1} = \frac{2.4 \text{ in.}}{2.0 \text{ in.}} = 1.2$$

$$K \approx 1.52 \qquad \tau_{max} = K \quad \tau_{mom} = K\left(\frac{16\,T_{max}}{\pi\,D_1^3}\right)$$

$$T_{max} = \frac{\pi D_1^3 \tau_{max}}{16K}$$

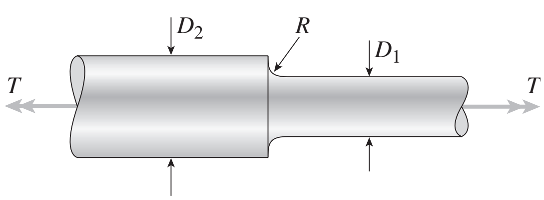
$$= \frac{\pi(2.0 \text{ in.})^3(6000 \text{ psi})}{16(1.52)} = 6200 \text{ lb-in.}$$

$$\therefore\ T_{max} \approx 6200 \text{ lb-in.} \longleftarrow$$

Problem 3.11-2 A stepped shaft with diameters $D_1 = 40$ mm and $D_2 = 60$ mm is loaded by torques $T = 1100\ \text{N}\cdot\text{m}$ (see figure).

If the allowable shear stress at the stress concentration is 120 MPa, what is the smallest radius R_{min} that may be used for the fillet?

Solution 3.11-2 **Stepped shaft in torsion**



$D_1 = 40$ mm

$D_2 = 60$ mm

$T = 1100\ \text{N}\cdot\text{m}$

$\tau_{allow} = 120$ MPa

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\pi D_1^3 \tau_{max}}{16T} = \frac{\pi(40 \text{ mm})^3(120 \text{ MPa})}{16(1100\ \text{N}\cdot\text{m})} = 1.37$$

$$\frac{D_2}{D_1} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$$

From Fig. (3-48) with $\frac{D_2}{D_1} = 1.5$ and $K = 1.37$,

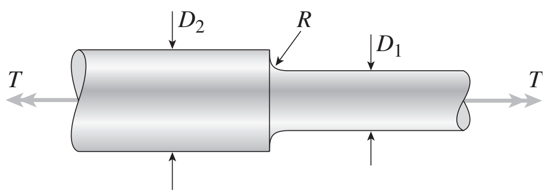
we get $\frac{R}{D_1} \approx 0.10$

$$\therefore\ R_{min} \approx 0.10(40 \text{ mm}) = 4.0 \text{ mm} \longleftarrow$$

Problem 3.11-3 A full quarter-circular fillet is used at the shoulder of a stepped shaft having diameter $D_2 = 1.0$ in. (see figure). A torque $T = 500$ lb-in. acts on the shaft.

Determine the shear stress τ_{max} at the stress concentration for values as follows: $D_1 = 0.7$, 0.8, and 0.9 in. Plot a graph showing τ_{max} versus D_1.

Solution 3.11-3 Stepped shaft in torsion



D_1 (in.)	D_2/D_1	R (in.)	R/D_1	K	τ_{max} (psi)
0.7	1.43	0.15	0.214	1.20	8900
0.8	1.25	0.10	0.125	1.29	6400
0.9	1.11	0.05	0.056	1.41	4900

$D_2 = 1.0$ in.
$T = 500$ lb-in.
$D_1 = 0.7$, 0.8, and 0.9 in.

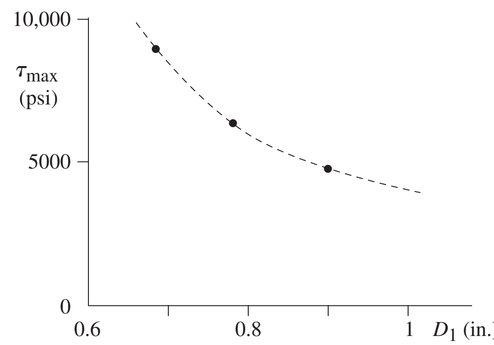
Full quarter-circular fillet ($D_2 = D_1 + 2R$)

$$R = \frac{D_2 - D_1}{2} = 0.5 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$= K\frac{16(500 \text{ lb-in.})}{\pi D_1^3} = 2546\frac{K}{D_1^3}$$

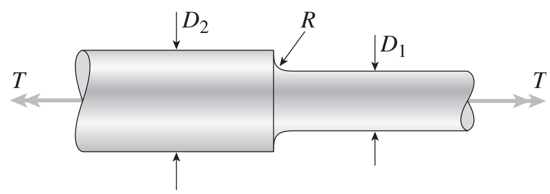


Note that τ_{max} gets smaller as D_1 gets larger, even though K is increasing.

Problem 3.11-4 The stepped shaft shown in the figure is required to transmit 600 kW of power at 400 rpm. The shaft has a full quarter-circular fillet, and the smaller diameter $D_1 = 100$ mm.

If the allowable shear stress at the stress concentration is 100 MPa, at what diameter D_2 will this stress be reached? Is this diameter an upper or a lower limit on the value of D_2?

Solution 3.11-4 **Stepped shaft in torsion**



$P = 600$ kW $\quad D_1 = 100$ mm

$n = 400$ rpm $\quad \tau_{allow} = 100$ MPa

Full quarter-circular fillet

POWER $P = \dfrac{2\pi n T}{60}$ (Eq. 3-42 of section 3.7)

$P =$ watts $\quad n =$ rpm $\quad T =$ Newton meters

$$T = \frac{60P}{2\pi n} = \frac{60(600 \times 10^3 \text{ W})}{2\pi(400 \text{ rpm})} = 14{,}320 \text{ N}\cdot\text{m}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\tau_{max}(\pi D_1^3)}{16T}$$

$$= \frac{(100 \text{ MPa})(\pi)(100 \text{ mm})^3}{16(14{,}320 \text{ N}\cdot\text{m})} = 1.37$$

Use the dashed line for a full quarter-circular fillet.

$$\frac{R}{D_1} \approx 0.075 \quad R \approx 0.075 \quad D_1 = 0.075 \ (100 \text{ mm})$$
$$= 7.5 \text{ mm}$$

$$D_2 = D_1 + 2R = 100 \text{ mm} + 2(7.5 \text{ mm}) = 115 \text{ mm}$$

$\therefore D_2 \approx 115$ mm ⟵

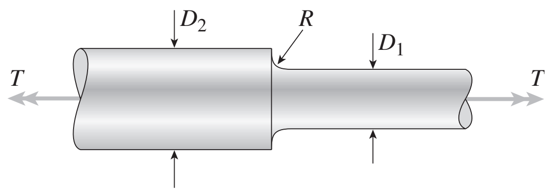
This value of D_2 is a *lower limit* ⟵

(If D_2 is less than 115 mm, R/D_1 is smaller, K is larger, and τ_{max} is larger, which means that the allowable stress is exceeded.)

Problem 3.11-5 A stepped shaft (see figure) has diameter $D_2 = 1.5$ in. and a full quarter-circular fillet. The allowable shear stress is 15,000 psi and the load $T = 4800$ lb-in.

What is the smallest permissible diameter D_1?

Solution 3.11-5 Stepped shaft in torsion



$D_2 = 1.5$ in.

$\tau_{allow} = 15{,}000$ psi

$T = 4800$ lb-in.

Full quarter-circular fillet $D_2 = D_1 + 2R$

$$R = \frac{D_2 - D_1}{2} = 0.75 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-48 FOR THE STRESS-CONCENTRATION FACTOR

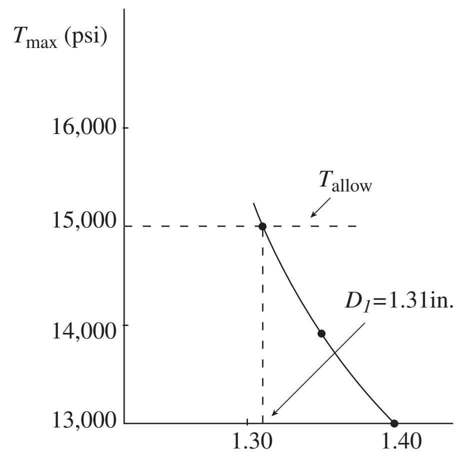
$$\tau_{max} = K\tau_{nom} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$= \frac{K}{D_1^3}\left[\frac{16(4800 \text{ lb-in.})}{\pi}\right]$$

$$= 24{,}450\,\frac{K}{D_1^3}$$

Use trial-and-error. Select trial values of D_1

D_1 (in.)	R (in.)	R/D_1	K	τ_{max}(psi)
1.30	0.100	0.077	1.38	15,400
1.35	0.075	0.056	1.41	14,000
1.40	0.050	0.036	1.46	13,000



From the graph, minimum $D_1 \approx 1.31$ in. ⟵